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On a question concerning zero sets of minimal area in domains of $\mathbb{C}^2$

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ABSTRACT

Let Ω be a convex domain in $\mathbb{C}^2$ symmetric with respect to the origin and let $f(z, w)$ run over the class of analytic functions on Ω which vanish at the origin. What is the minimal area of a zero set V of such a function f in Ω? In view of known results for the ball, a cube and tube domains, cf. the preceding paper, one might conjecture that the minimum is always attained only if V is a suitable linear variety. The present paper shows, however, that a linear variety V can minimize area only if a rather special analyticity condition is satisfied. The latter condition on Ω and V makes it possible to construct counterexamples to the conjecture.

1. INTRODUCTION

Let Ω be a convex domain in $\mathbb{C}^2$ containing the origin O. Let V be the zero set of a holomorphic function on Ω which vanishes at O but is not identically zero. Suppose that V has minimal area with respect to all zero varieties in Ω of this kind. It is a consequence of a theorem of Bishop (theorem (C) of [9]) that such an area minimizing variety always exists (and not just for convex domains). In the following three cases it is known that the varieties V of minimal area must be linear:

(i) If Ω is a ball [5, 8, 1];

(ii) If Ω is a cube centered at O and with edges parallel to the coordinate axes [3];

(iii) If Ω is a (convex) tube domain symmetric with respect to O (see the preceding paper [4]).

Case (iii), as well as case (i) if O is not the center of the ball, show that a linear variety of minimal area need not intersect the boundary of Ω at right angles as one would perhaps expect. The question arises whether (i), (ii) and (iii) are just special cases of a rather general theorem. For example, what one could hope for is the validity of the following:

BOLD CONJECTURE. *If Ω and the area minimizing variety V are as above, and if Ω is in addition assumed to be symmetric, then V is linear.*

Let us note that the conjecture is true if "symmetric" is replaced by "circular" (i.e. if $(z, w) \in \Omega$ then $(e^{i\theta}z, e^{i\theta}w) \in \Omega$ for all real θ). In fact this follows immediately from case (i).

The aim of this paper is to show that a linear variety V can minimize area only if an interesting analyticity condition involving Ω and V is satisfied. This result will in particular disprove the above conjecture: we will give a simple explicit example of a smoothly bounded convex symmetric domain in which no linear variety through the origin minimizes area. Our theorem shows that, to the contrary, cases such as (i), (ii) and (iii) must be rather exceptional.

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2. THE RESULT: AN ANALYTICITY CONDITION FOR LINEAR VARIETIES OF MINIMAL AREA

In the following Ω is assumed to be a bounded convex domain in $\mathbb{C}^2$ with C^2-boundary which contains the origin O. Let ϱ be a defining function for Ω, i.e. ϱ is a real-valued C^2-function on $\mathbb{C}^2$ such that

$$\Omega = \{\varrho < 0\}, \ \nabla\varrho \neq 0 \text{ on } \partial\Omega.$$

Finally, let V be the restriction to Ω of a linear plane

$$az + bw = 0, \ |a|^2 + |b|^2 = 1,$$

and let ∂V (abusively) denote its boundary curve (which is also convex and of class C^2).

Further notations to be used here and in the sequel will be as in [4]. In particular elements of $\mathbb{C}^2$ are denoted by

$$(z, w) = (x + iy, u + iv)$$

and Ar denotes area.

THEOREM. *Let Ω, ϱ and the linear variety V be as above. Suppose that V has minimal area among all zero sets (restricted to Ω) of polynomials in z and w which vanish at O. Then the function*

$$g := \frac{\bar{a}\dfrac{\partial\varrho}{\partial z} + \bar{b}\dfrac{\partial\varrho}{\partial w}}{\bar{b}\dfrac{\partial\varrho}{\partial\bar{z}} - \bar{a}\dfrac{\partial\varrho}{\partial\bar{w}}} \tag{1}$$

restricted to ∂V admits a continuous extension to $\bar{V}-\{O\}$ which is holomorphic on $V-\{O\}$ and has at most a simple pole at the origin. (*Of course these notions refer to the obvious complex structure on V.*)

REMARK 1. Replacement of ϱ by $\varphi \cdot \varrho$ where φ is an everywhere positive C^1-function does not alter the function g on ∂V. Therefore g does not depend on the particular choice of the defining function ϱ.

REMARK 2. Note that g is well-defined on ∂V, i.e. the denominator in (1) does not vanish on ∂V. We need verify this only when $a=0$, $b=1$ (cf. the proof in section 4). If now $(z_0, 0)$ is a point of $\partial \Omega$ with $\partial \varrho/\partial z=0$ it would follow that the complex tangent plane to $\partial \Omega$ at $(z_0, 0)$ is given by $w=0$. But this is impossible since Ω is convex and $O \in \Omega$. (This is essentially the only place where the convexity of Ω is used.)

3. DISCUSSION OF THE RESULT

Let us look first what g becomes for the minimal planes in balls and tube domains (cf. cases (i), (iii) of section 1). When Ω is a ball $|z|^2+|w|^2<R^2$ the function g is identically zero. (In fact this holds for arbitrary Ω when the linear variety V intersects $\partial\Omega$ at right angles. In particular the necessary condition expressed by the theorem is not a sufficient one.) When Ω is a "non-symmetric" ball

$$|z|^2+|w-w_0|^2<R^2 \ (0<|w_0|<R)$$

the (unique) minimal variety through O is the plane $w=0$ (cf. [1]). Here

$$g=g(z)=-\frac{\bar{w}_0}{z}.$$

This example shows that a simple pole at O can actually occur.

Finally let Ω be a convex symmetric tube domain (with C^2-boundary). In this case the minimal varieties through O are $z+iw=0$ and $z-iw=0$ (cf. [4]). We find $g \equiv i$ and $g \equiv -i$, respectively. Indeed, in this case we may assume that $\varrho(z, w)=\varrho(x, u)$, so that $\partial\varrho/\partial z=\partial\varrho/\partial\bar{z}$, $\partial\varrho/\partial w=\partial\varrho/\partial\bar{w}$. (We need not worry about the fact that the theorem is stated for bounded domains only. Actually our proof remains valid under much weaker hypotheses on Ω and V than stated. In particular the proof will apply to tube domains and the planes $z \pm iw=0$.)

On the other hand the expression for g is so non-analytic that for a "generic" domain Ω no linear variety will minimize area. It is easy to give an explicit

Counterexample to the conjecture of section 1. For $\varepsilon>0$ let $\Omega=\Omega_\varepsilon$ be the domain with defining function

(2) $$\varrho(z, w)=|z|^2+|w|^2+\varepsilon\alpha(z, w)-1,$$

where

$$\alpha(z, w)=\begin{cases} (xy)^3|w|^2 & \text{if } xy>0, \\ 0 & \text{if } xy \leq 0. \end{cases}$$

The domain Ω_ε is properly contained in the unit ball and coincides with it in the regions where $xy \leq 0$. Furthermore Ω_ε will be a (strictly) convex symmetric domain with C^2-boundary, at least when ε is not too large. If V is the linear plane $az+bw=0$ we find that

$$\left(\bar{a}\frac{\partial\varrho}{\partial z}+\bar{b}\frac{\partial\varrho}{\partial w}\right)\bigg|_{\partial V}=\varepsilon\left(\bar{a}\frac{\partial\alpha}{\partial z}+\bar{b}\frac{\partial\alpha}{\partial w}\right)\bigg|_{\partial V},$$

hence g vanishes on the part of ∂V lying in the set $xy \leq 0$. However, an easy calculation shows that g does not vanish identically on ∂V unless $a=0$ or $b=0$. Therefore g is not the (continuous) boundary value of a holomorphic function, except when $a=0$ or $b=0$. (A non-zero bounded holomorphic function on a smooth domain in $\mathbb{C}$ can not have boundary value zero on a boundary arc of positive length.) But if $a=0$ or $b=0$ we have Ar $V=\pi$, so again V does not have minimal area!

It is easy to obtain counterexamples with C^∞-boundary, e.g. by replacing the factor $(xy)^3$ in the definition of α by $\exp(-1/xy)$.

The proof of the theorem will be given in two parts.

4. PROOF OF THE THEOREM (FIRST PART)

With the aid of the unitary transformation

$$\begin{matrix} z'=\bar{b}z-\bar{a}w & & z=bz'+\bar{a}w' \\ & \text{or} & \\ w'=az+bw & & w=-az'+\bar{b}w' \end{matrix}$$

it follows easily that we may restrict ourselves to the case $a=0$, $b=1$, i.e.

$$V=\{(z,w)\in\Omega \mid w=0\}.$$

We then have to prove that the function

$$zg(z)=z\frac{\partial\varrho}{\partial w}(z,0)\Big/\frac{\partial\varrho}{\partial\bar{z}}(z,0)$$

restricted to ∂V has a continuous extension to $\bar{V}$ which is analytic on V (regarded as a plane domain in $\mathbb{C}$). Such an extension will exist if (and by Cauchy's theorem only if)

(3) $$\int_{\partial V} z^n g(z)\,dz=0,\ n=1,2,\ldots.$$

Indeed, the Cauchy transform

$$\frac{1}{2\pi i}\int_{\partial V}\frac{\zeta g(\zeta)}{\zeta-z}\,d\zeta$$

vanishes identically outside $\bar{V}$ if (3) holds. By the theory around the *Plemelj formula* for the jump of a Cauchy-integral, this transform must therefore have continuous boundary values equal to $zg(z)$ under approach to ∂V from the inside of V. (Cf. [6] and also [7], pp. 136 and 141, 142; for another proof based on conformal mapping and Walsh's approximation theorem, cf. [2], theorems 10.4, 10.5.)

We shall prove that (3) holds. So let us fix a positive integer n. For complex λ we consider the zero sets

$$V_\lambda=\{(z,w)\in\Omega \mid w=\lambda z^n\}$$

and we denote by V_λ^z and V_λ^w their z- and w-projections, respectively. Note that $V=V_0=V_0^z$, so by our assumption on V we have for all λ

$$\text{(4)} \qquad \operatorname{Ar} V_0^z=\operatorname{Ar} V_0\leq \operatorname{Ar} V_\lambda.$$

As is well-known, the area of V_λ is equal to the sum of the areas of the projections V_λ^z and V_λ^w (Wirtinger [10]). It follows that

$$\operatorname{Ar} V_\lambda=\operatorname{Ar} V_\lambda^z+O(|\lambda|^2),$$

since V_λ^w is obviously contained in a disc of radius bounded by $C|\lambda|$ while each point of V_λ^w has at most n points of V_λ over it. (More precisely we have

$$\operatorname{Ar} V_\lambda= \int (1+|n\lambda z^{n-1}|^2)dxdy,$$

where the integration is over those z for which $(z,\lambda z^n)\in\Omega$.) Thus we obtain from (4)

$$\text{(5)} \qquad \operatorname{Ar} V_0^z\leq \operatorname{Ar} V_\lambda^z+O(|\lambda|^2).$$

This holds in particular for $\lambda=t$ and $\lambda=it$, with t real. If we can show that $\operatorname{Ar} V_t^z$ and $\operatorname{Ar} V_{it}^z$ are differentiable at $t=0$, it will follow from (5) that

$$\text{(6)} \qquad \frac{\partial}{\partial t}\operatorname{Ar} V_t^z\Big|_{t=0}=\frac{\partial}{\partial t}\operatorname{Ar} V_{it}^z\Big|_{t=0}=0.$$

Before we can continue with the proof we have to take a close look at the differentiation of integrals of a certain type.

5. DIFFERENTIATION OF INTEGRALS OVER VARIABLE REGIONS

LEMMA. *Let $\tilde{\varrho}(z,t)$ be a real-valued C^1-function defined on $\mathbb{C}\times(a,b)$. Suppose that $\partial\tilde{\varrho}/\partial z\neq 0$ when $\tilde{\varrho}=0$ and that the open sets*

$$G_t=\{z\in\mathbb{C} \mid \tilde{\varrho}(z,t)<0\}$$

lie within a fixed bounded set for $t\in(a,b)$. Let f be a continuous function on $\mathbb{C}$ and define

$$F(t)=\int_{G_t} f\,dxdy,\quad t\in(a,b).$$

Then F is differentiable and

$$\text{(7)} \qquad F'(t)=\tfrac{1}{2}i\int_{\partial G_t} f(z)\,\frac{\partial\tilde{\varrho}}{\partial t}(z,t)\Big/\frac{\partial\tilde{\varrho}}{\partial\bar{z}}(z,t)\,dz.$$

PROOF. We may assume that $0\in(a,b)$ and will prove (7) for $t=0$. The assumption that all G_t lie in a fixed bounded set implies that if U is an arbitrarily small neighbourhood of $\bar{G}_0$, then G_t belongs to U as soon as $|t|$ is sufficiently small. It follows from this and a simple partition of unity argument that it suffices to consider the following two cases:

(a) f has compact support in G_0;

(b) f has compact support in an arbitrarily small neighbourhood Q (which will be specified later) of some point z_0 on ∂G_0.

Case (a) is trivial: if f has compact support in G_0 it will also have compact support in G_t if $|t|$ is sufficiently small. So let us concentrate on case (b). We may take $z_0=0$. By assumption then $\partial\tilde{\varrho}/\partial z\neq 0$ at $(0,0)$. A simple rotation allows us to assume further that $\partial\tilde{\varrho}/\partial y>0$ at $(0,0)$. By the implicit function theorem we get for small $|x|$ and $|t|$ a unique C^1-function $\varphi(x,t)$ satisfying $\varphi(0,0)=0$ and

$$\tilde{\varrho}(x+i\varphi(x,t),t)\equiv 0.$$

For $\delta>0$ let $Q=Q_\delta$ be the square $\{z\in\mathbb{C}\,|\,|x|<\delta,\ |y|<\delta\}$. If $\delta>0$ is small enough we will have $(\partial\tilde{\varrho}/\partial y)(z,0)>0$ on Q and

$$G_t\cap Q=\{z\in Q\,|\,y<\varphi(x,t)\}$$

for all small $|t|$.

Now assume (we are in case (b)) that the support of f is contained in Q. Fubini's theorem and the uniform continuity of f on ∂G_0 then give as $t\to 0$:

$$\text{(8)}\qquad \frac{F(t)-F(0)}{t}=\int_{-\delta}^{\delta}\left\{\frac{1}{t}\int_{\varphi(x,0)}^{\varphi(x,t)} f(x+iy)\,dy\right\}dx\to\int_{-\delta}^{\delta} f(x+i\varphi(x,0))\,\frac{\partial\varphi}{\partial t}(x,0)\,dx.$$

Hence F is differentiable at $t=0$.

Let us evaluate the right hand side of (8). First note that

$$\frac{\partial\varphi}{\partial x}=-\frac{\partial\tilde{\varrho}}{\partial x}\Big/\frac{\partial\tilde{\varrho}}{\partial y},\qquad \frac{\partial\varphi}{\partial t}=-\frac{\partial\tilde{\varrho}}{\partial t}\Big/\frac{\partial\tilde{\varrho}}{\partial y}.$$

It follows that on $\partial G_t\cap Q$ we have, ignoring the orientation,

$$dz=d(x+i\varphi(x,t))=\left(1+i\,\frac{\partial\varphi}{\partial x}\right)dx$$

$$=\left(\frac{\partial\tilde{\varrho}}{\partial y}-i\,\frac{\partial\tilde{\varrho}}{\partial x}\right)\Big/\frac{\partial\tilde{\varrho}}{\partial y}\,dx=-2i\,\frac{\partial\tilde{\varrho}}{\partial\bar{z}}\Big/\frac{\partial\tilde{\varrho}}{\partial y}\,dx.$$

Consequently

$$\text{(9)}\qquad \frac{\partial\varphi}{\partial t}\,dx=-\frac{\partial\tilde{\varrho}}{\partial t}\Big/\frac{\partial\tilde{\varrho}}{\partial y}\,dx=-\tfrac{1}{2}i\,\frac{\partial\tilde{\varrho}}{\partial t}\Big/\frac{\partial\tilde{\varrho}}{\partial\bar{z}}\,dz.$$

We use (9) for $t=0$ and observe that the right hand side of (8) would correspond to clockwise traversal of ∂G_0. Changing to the normal positive orientation we finally see that

$$F'(0)=\tfrac{1}{2}i\int_{\partial G_0} f(z)\,\frac{\partial\tilde{\varrho}}{\partial t}(z,0)\Big/\frac{\partial\tilde{\varrho}}{\partial\bar{z}}(z,0)\,dz,$$

as was to be proved.

6. PROOF OF THE THEOREM (SECOND PART)

We can now finish the proof of the theorem. Note that $\varrho(z, \lambda z^n)$ is a C^1-defining function for V_λ^z if $|\lambda|$ is small enough (recall that $(\partial\varrho/\partial z)(z,0) \neq 0$ if $\varrho(z,0)=0$, cf. remark 2 in section 2). Furthermore the domains V_λ^z are contained in a fixed bounded set since Ω is bounded. The lemma applies therefore with

$$\tilde{\varrho}(z,t) = \varrho(z, tz^n)$$

and with $f \equiv 1$. Hence we obtain

$$(10)\quad \left\{ \begin{aligned} \frac{\partial}{\partial t} \operatorname{Ar} V_t^z \Big|_{t=0} &= \tfrac{1}{2} i \int_{\partial V} \left(\frac{\partial \varrho}{\partial w} z^n + \frac{\partial \varrho}{\partial \bar{w}} \bar{z}^n \right) \Big/ \frac{\partial \varrho}{\partial \bar{z}}\, dz \\ &= i \int_{\partial V} \operatorname{Re} \left(z^n \frac{\partial \varrho}{\partial w} \right) \Big/ \frac{\partial \varrho}{\partial \bar{z}}\, dz. \end{aligned} \right.$$

A similar application of the lemma with $\tilde{\varrho}(z,t) = \varrho(z, itz^n)$ gives

$$(11)\quad \frac{\partial}{\partial t} \operatorname{Ar} V_{it}^z \Big|_{t=0} = -i \int_{\partial V} \operatorname{Im} \left(z^n \frac{\partial \varrho}{\partial w} \right) \Big/ \frac{\partial \varrho}{\partial \bar{z}}\, dz.$$

Together (6), (10) and (11) imply

$$\int_{\partial V} z^n \frac{\partial \varrho}{\partial w} \Big/ \frac{\partial \varrho}{\partial \bar{z}}\, dz = 0.$$

Since $n \geq 1$ was arbitrary, (3) and hence the theorem are now completely proved.

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